DAY SIXTEEN

Definite Integrals

Learning & Revision for the Day

- Concept of Definite Integrals
- Leibnitz Theorem
- Walli's Formula
- Inequalities in Definite Integrals
- Definite Integration as the Limit of a sum

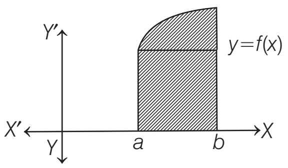
Concept of Definite Integrals

Let $\phi(x)$ be an anti-derivative of a function $f(x)$ defined on $[a,b]$ i.e. $\frac{d}{dx}[\phi(x)] = f(x)$. Then, definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$ i.e. $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$. The numbers a and b are called the limits of integration, where a is called **lower limit** and b is **upper limit**.

NOTE
- Every definite integral has a unique value.
- The above definition is nothing but the statement of second fundamental theorem of integral calculus.

Geometrical Interpretation of Definite Integral

In general, $\int_a^b f(x)dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$ as show in the following figure.



Evaluation of Definite Integrals by Substitution

When the variable in a definite integral is changed due to substitution, then the limits of the integral will accordingly be changed.

For example, to evaluate definite integral of the form

$\int_a^b f[g(x)] \cdot g'(x)\,dx$, we use the following steps

***Step* I** Substitute $g(x) = t$ so that $g'(x)\,dx = dt$

***Step* II** Find the limits of integration in new system of variable. Here, the lower limit is $g(a)$, the upper limit is $g(b)$ and the integral is now $\int_{g(a)}^{g(b)} f(t)\,dt$.

***Step*III** Evaluate the integral, so obtained by usual method.

Properties of Definite Integrals

(i) $\int_\alpha^\alpha f(x)dx = 0$

(ii) $\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(t)\,dt$

(iii) $\int_\alpha^\beta f(x)\,dx = -\int_\beta^\alpha f(x)\,dx$

(iv) $\int_\alpha^\beta f(x)\,dx = \int_\alpha^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_n}^\beta f(x)\,dx$

where, $\alpha < c_1 < c_2 < \ldots < c_n < \beta$

(v) (a) $\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(\alpha + \beta - x)\,dx$

(b) $\int_0^\alpha f(x)\,dx = \int_0^\alpha f(\alpha - x)\,dx$

(vi) $\int_{-\alpha}^\alpha f(x)\,dx$

$$= \begin{cases} 2\int_0^\alpha f(x)\,dx, \text{ if } f(-x) = f(x) \\ \qquad \text{i.e. } f(x) \text{ is an even function} \\ 0, \quad \text{if } f(-x) = -f(x) \\ \qquad \text{i.e. } f(x) \text{ is an odd function} \end{cases}$$

(vii) $\int_0^{2\alpha} f(x)dx = \int_0^\alpha f(x)dx + \int_0^\alpha f(2\alpha - x)dx$

(viii) $\int_0^{2\alpha} f(x)\,dx = \begin{cases} 2\int_0^\alpha f(x)\,dx, & \text{if } f(2\alpha - x) = f(x) \\ 0, & \text{if } f(2\alpha - x) = -f(x) \end{cases}$

(ix) $\int_\alpha^\beta f(x)\,dx = (\beta - \alpha)\int_0^1 f[(\beta - \alpha)\,x + \alpha]dx$

(x) If $f(x)$ is a periodic function with period T, then

(a) $\int_\alpha^{\alpha + nT} f(x)\,dx = n\int_0^T f(x)\,dx, n \in I$

(b) $\int_{\alpha T}^{\beta T} f(x)\,dx = (\beta - \alpha)\int_0^T f(x)\,dx, \quad \alpha, \beta \in I$

(c) $\int_{\alpha + nT}^{\beta + nT} f(x)\,dx = \int_\alpha^\beta f(x)\,dx, n \in I$

(xi) Some important integrals, which can be obtained with the help of above properties.

(a) $\int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right).3$

(b) $\int_0^{\pi/4} \log(1 + \tan x)dx = \frac{\pi}{8}\log 2.$

(xii) If a function $f(x)$ is discontinuous at points $x_1, x_2, \ldots, x_n$ in (a, b), then we can define sub-intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, b)$ such that $f(x)$ is continuous in each of these sub-intervals and

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx + \int_{x_n}^b f(x)dx.$$

Leibnitz Theorem

If function $\phi(x)$ and $\psi(x)$ are defined on $[\alpha, \beta]$ and differentiable on $[\alpha, \beta]$ and $f(t)$ is continuous on $[\psi(\alpha), \phi(\beta)]$, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = \left\{\frac{d}{dt}\,\psi(x)\right\} f(\psi(x)) - \left\{\frac{d}{dx}\{\phi(x)\}\right\}\{f(\phi(x)\}$$

Walli's Formula

This is a special type of integral formula whose limits from 0 to $\pi/2$ and integral is either integral power of $\cos x$ or $\sin x$ or $\cos x \sin x$.

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 5\cdot 3\cdot 1}{n(n-2)(n-4)\ldots 6\cdot 4\cdot 2} \times \dfrac{\pi}{2}, & \text{if } n = 2m \text{ (even)} \\ \dfrac{(n-1)(n-3)(n-5)\ldots 6\cdot 4\cdot 2}{n(n-2)(n-4)\ldots 5\cdot 3\cdot 1}, & \text{if } n = 2m + 1 \text{ (odd)} \end{cases}$$

where, n is positive integer.

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$$

$$= \begin{cases} \dfrac{(m-1)(m-3)\ldots(2 \text{ or } 1).(n-1)(n-3)\ldots(2 \text{ or } 1)}{(m+n)(m+n-2)\ldots(2 \text{ or } 1)}\dfrac{\pi}{2}, \\ \text{when both } m \text{ and } n \text{ are even positive integers} \\ \dfrac{(m-1)(m-3)\ldots(2 \text{ or } 1)\cdot(n-1)(n-3)\ldots(2 \text{ or } 1)}{(m+n)(m+n-2)\ldots(2 \text{ or } 1)}, \\ \text{when either } m \text{ or } n \text{ or both are odd} \\ \text{positive integers} \end{cases}$$

Inequalities in Definite Integrals

(i) If $f(x) \ge g(x)$ on $[\alpha, \beta]$, then $\int_\alpha^\beta f(x)\,dx \ge \int_\alpha^\beta g(x)\,dx$

(ii) If $f(x) \ge 0$ in the interval $[\alpha, \beta]$, then $\int_\alpha^\beta f(x)\,dx \ge 0$

(iii) If $f(x), g(x)$ and $h(x)$ are continuous on $[a, b]$ such that $g(x) \le f(x) \le h(x)$, then $\int_a^b g(x)\,dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$

(iv) If f is continuous on $[\alpha, \beta]$ and $l \le f(x) \le M, \forall$ $x \in [\alpha, \beta]$, then $l(\beta - \alpha) \le \int_\alpha^\beta f(x)\,dx \le M(\beta - \alpha)$

(v) If f is continuous on $[\alpha, \beta]$, then $\left|\int_\alpha^\beta f(x)\,dx\right| \le \int_\alpha^\beta |f(x)|\,dx$

(vi) If f is continuous on $[\alpha, \beta]$ and $|f(x)| \le k, \forall x \in [\alpha, \beta]$, then $\left|\int_\alpha^\beta f(x)\,dx\right| \le k(\beta - \alpha)$

Definite Integration as the Limit of a Sum

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a + rh)$

where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

The converse is also true, i.e. if we have an infinite series of the above form, it can be expressed as definite integral.

Some Particular Cases

(i) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$ or $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

(ii) $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{h} = 0$ $(\because r = 1)$

and $\beta = \lim_{n\to\infty} \frac{r}{h} = p$ $(\because r = pn)$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to → **JEE Mains 2017**

(a) -2 (b) 2 (c) 4 (d) -1

2 If $f(x)$ is continuous function, then

(a) $\int_{-2}^{2} f(x)\,dx = \int_0^2 [f(x) - f(-x)]\,dx$

(b) $\int_{-3}^{5} 2f(x)\,dx = \int_6^{10} f(x-1)\,dx$

(c) $\int_{-3}^{5} f(x)\,dx = \int_{-4}^{4} f(x-1)\,dx$

(d) $\int_{-3}^{5} f(x)\,dx = \int_{-2}^{6} f(x-1)\,dx$

3 $\int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}]\,dx$ is equal to

(a) $\sqrt{2}\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{\sqrt{2}}$ (d) 2π

4 $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)dx$ is equal to

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/2$ (d) π

5 If $I_{1(n)} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)}{\sin x}\,dx$ and $I_{2(n)} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x}\,dx$, $n \in N$, then

(a) $I_{2(n+1)} - I_{2(n)} = I_{1(n)}$ (b) $I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$
(c) $I_{2(n+1)} + I_{1(n)} = I_{2(n)}$ (d) $I_{2(n+1)} + I_{1(n+1)} = I_{2(n)}$

6 $\int_{-1}^{1} \{x^2 + x - 3\}\,dx$, where $\{x\}$ denotes the fractional part of x, is equal to

(a) $\frac{1}{3}(1 + 3\sqrt{5})$ (b) $\frac{1}{6}(1 + 3\sqrt{5})$

(c) $\frac{1}{3}(3\sqrt{5} - 1)$ (d) $\frac{1}{6}(3\sqrt{5} - 1)$

7 $\int_0^2 [x^2]\,dx$ is equal to

(a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{2} - 1$ (d) $-\sqrt{2} - \sqrt{3} + 5$

8 If $\int_{-2}^{x} |2t|\,dt = f(x)$, then for any $x \ge 0, f(x)$ is equal to

(a) $4 + x^2$ (b) $4 - x^2$
(c) $\frac{1}{2}(4 + x^2)$ (d) $\frac{1}{4}(4 - x^2)$

9 $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x}\,dx$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

10 Let a, b and c be non-zero real numbers such that $\int_0^3 (3ax^2 + 2bx + c)\,dx = \int_1^3 (3ax^2 + 2bx + c)\,dx$, then

(a) $a + b + c = 3$ (b) $a + b + c = 1$
(c) $a + b + c = 0$ (d) $a + b + c = 2$

11 The value of $\int_1^a [x] f'(x)\,dx, a > 1$, where $[x]$ denotes the greatest integer not exceeding x, is

(a) $[a]f(a) - \{f(1) + f(2) + \ldots + f([a])\}$
(b) $[a]f([a]) - \{f(1) + f(2) + \ldots + f(a)\}$
(c) $af([a]) - \{f(1) + f(2) + \ldots + f(a)\}$
(d) $af(a) - \{f(1) + f(2) + \ldots + f([a])\}$

12 The correct evaluation of $\int_0^{\pi/2} \left|\sin\left(x - \frac{\pi}{4}\right)\right| dx$ is

(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$
(c) $-2 + \sqrt{2}$ (d) 0

13 $\int_0^{3\pi/2} \sin\left(\left[\frac{2x}{\pi}\right]\right) dx$, where [·] denotes the greatest integer function is equal to

(a) $\frac{\pi}{2}(\sin 1 + \cos 1)$ (b) $\frac{\pi}{2}(\sin 1 - \sin 2)$

(c) $\frac{\pi}{2}(\sin 1 - \cos 1)$ (d) $\frac{\pi}{2}(\sin 1 + \sin 2)$

14 If [.] denotes the greatest integer function, then the value of $\int_0^{1.5} x[x^2]\,dx$ is

(a) $\frac{5}{4}$ (b) 0 (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

15 The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2} dx$ is

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{2}\log 2$ (c) $\log 2$ (d) $\pi\log 2$

16 $\int_0^{\pi/2} \sqrt{1-\sin 2x}$ is equal to ➜ **NCERT Exemplar**

(a) $2\sqrt{2}$ (b) $2(\sqrt{2}+1)$ (c) 2 (d) $2(\sqrt{2}-1)$

17 The value of $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is

(a) 3/2 (b) 5/2 (c) 3 (d) 5

18 The value of $\sum_{k=1}^{n} \int_0^1 f(k-1+x)\,dx$ is

(a) $\int_0^1 f(x)\,dx$ (b) $\int_0^2 f(x)\,dx$ (c) $\int_0^n f(x)\,dx$ (d) $n\int_0^1 f(x)\,dx$

19 $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x)\,dx = \frac{n^2}{2}, \forall n \in I$, then $\int_{-3}^{5} f(|x|)\,dx$ is equal to

(a) $\frac{19}{2}$ (b) $\frac{35}{2}$

(c) $\frac{17}{2}$ (d) None of these

20 If $I_1 = \int_0^1 2^{x^2} dx, I_2 = \int_0^1 2^{x^3} dx, I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$, then

(a) $I_3 > I_4$ (b) $I_3 = I_4$
(c) $I_1 > I_2$ (d) $I_2 > I_1$

21 The value of $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\,dx$ is ➜ **JEE Mains 2013**

(a) $\log 2\sqrt{2}$ (b) $\frac{3}{2}\log 2$

(c) $2\log 2$ (d) $\log\sqrt{2}$

22 $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ is equal to ➜ **NCERT Exemplar**

(a) $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ (b) $\frac{1}{2}\log(\sqrt{2}+1)$

(c) $-\log(\sqrt{2}+1)$ (d) None of these

23 The value of $\lim_{n\to\infty} \frac{1}{n}\left\{\frac{1}{n+1} + \frac{2}{n+2} + \ldots + \frac{3n}{4n}\right\}$ is

(a) $5 - 2\ln 2$ (b) $4 - 2\ln 2$
(c) $3 - 2\ln 2$ (d) $2 - 2\ln 2$

24 If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x)\cdot g(x)\,dx$ is equal to ➜ **NCERT Exemplar**

(a) $\frac{a}{2}$ (b) $\frac{a}{2}\int_0^a f(x)\,dx$

(c) $\int_0^a f(x)\,dx$ (d) $a\int_0^a f(x)\,dx$

25 The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ is ➜ **JEE Mains 2018, 13**

(a) π (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

26 $\int_0^{\pi} [\cot x]\,dx$, where [] denotes the greatest integer function, is equal to

(a) $\frac{\pi}{2}$ (b) 1 (c) -1 (d) $-\frac{\pi}{2}$

27 The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to ➜ **JEE Mains 2015**

(a) 2 (b) 4 (c) 1 (d) 6

28 $\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)]\,dx$ is equal to

(a) $\left(\frac{\pi^4}{32}\right) + \frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\left(\frac{\pi}{4}\right) - 1$ (d) $\frac{\pi^4}{32}$

29 If $f: R \to R$ and $g: R \to R$ are one to one, real valued function, then the value of the integral $\int_{-\pi}^{\pi} [f(x) + f(-x)][g(x) - g(-x)]\,dx$ is

(a) 0 (b) π (c) 1 (d) None of these

30 The value of $\lim_{n\to\infty}\left(\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \frac{n}{(n+3)\sqrt{3(2n+3)}} + \ldots + \frac{1}{2n\sqrt{3}}\right)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) None of these

31 If $P = \int_0^{3\pi} f(\cos^2 x)\,dx$ and $Q = \int_0^{\pi} f(\cos^2 x)\,dx$, then

(a) $P - Q = 0$ (b) $P - 2Q = 0$
(c) $P - 3Q = 0$ (d) $P - 5Q = 0$

32 $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$ is equal to

(a) $\frac{\pi^2}{4}$ (b) π^2 (c) 0 (d) $\frac{\pi}{2}$

33 If $f(x)$ is differentiable and $\int_0^{t^2} x\, f(x)\, dx = \frac{2}{5} t^5$, then $f\left(\frac{4}{25}\right)$ is equal to

(a) $\frac{2}{5}$ (b) $-\frac{5}{2}$ (c) 1 (d) $\frac{5}{2}$

34 If $f(x) = \frac{1}{x^2} \int_4^x [4t^2 - 2f'(t)]\, dt$, then $f'(4)$ is equal to

(a) 32 (b) $\frac{32}{3}$
(c) $\frac{32}{9}$ (d) None of these

35 The value of $\lim_{x \to 0} \frac{\int_0^{x^2} \sin\sqrt{t}\, dt}{x^3}$ is

(a) 0 (b) 2 / 9
(c) 1/ 3 (d) 2 / 3

36 If $I_1 = \int_1^2 \frac{dx}{\sqrt{1+x^2}}$ and $I_2 = \int_1^2 \frac{dx}{x}$, then

(a) $I_1 > I_2$ (b) $I_2 > I_1$ (c) $I_1 = I_2$ (d) $I_1 > 2\, I_2$

37 If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to

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(a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2

38 Let $f: R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \left(\frac{1}{48}\right)$. Then, $\lim_{x \to 2} \int_6^{f(x)} \frac{4t^3}{x-2}\, dt$ is equal to

(a) 18 (b) 12 (c) 36 (d) 24

39 $\lim_{n \to \infty} \frac{[1 + 2^4 + 3^4 + \ldots + n^4]}{n^5} - \frac{[1 + 2^3 + 3^3 + \ldots + n^3]}{n^5}$ is equal to

(a) $\frac{1}{30}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

40 If $I = \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}}\, dx$, then

(a) $0 < I < 1$ (b) $I > \frac{\pi}{2\sqrt{2}}$
(c) $I < \sqrt{2}\, \pi$ (d) $I < 2\pi$

41 If $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\, dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$. Then, which one of the following is true?

(a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$
(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

42 **Statement I** $\int_0^2 f(x)\, dx = \frac{4(\sqrt{2}-1)}{3}$,

where, $f(x) = \begin{cases} x^2, & \text{for } 0 \le x < 1 \\ \sqrt{x}, & \text{for } 1 \le x \le 2 \end{cases}$

Statement II $f(x)$ is continuous in [0, 2].

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for statement I
(c) Statement I is true; Statement II is false
(d)Statement I is false; Statement II is true

43 **Statement I** The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$ is $\frac{\pi}{6}$.

Statement II $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

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(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d)Statement I is false; Statement II is true

44 **Statement I** If $\int_0^1 e^{\sin x} dx = \lambda$, then $\int_0^{200} e^{\sin x} dx = 200\lambda$

Statement II If $\int_0^{na} f(x)\, dx = n \int_0^a f(x)\, dx$, $n \in I$ and $f(a + x) = f(x)$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true, Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d)Statement I is false; Statement II is true

45 **Statement I** $\int_0^1 e^{-x} \cos^2 x\, dx < \int_0^1 e^{-x^2} \cos^2 x\, dx$

Statement II $\int_a^b f(x)\, dx < \int_a^b g(x)\, dx, \forall f(x) < g(x)$

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true, Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d)Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right)+x^2 f(x)=0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx$ is equal to

(a) $\sin\theta+\operatorname{cosec}\theta$ (b) $\sin^2\theta$
(c) $\operatorname{cosec}^2\theta$ (d) None of these

2 If $\frac{d}{dx}f(x)=\frac{e^{\sin x}}{x}$; $x>0$. If $\int_1^4 \frac{3}{x}e^{\sin x^3}\,dx=f(k)-f(1)$, then the possible value of k, is

(a) 15 (b) 16 (c) 63 (d) 64

3 If $g(1)=g(2)$, then $\int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\}\,g'(x)\,dx$ is equal to

(a) 1 (b) 2 (c) 0 (d) None of these

4 For $0\le x\le\frac{\pi}{2}$, the value of $\int_0^{\sin^2 x}\sin^{-1}(\sqrt{t})\,dt+\int_0^{\cos^2 x}\cos^{-1}(\sqrt{t})\,dt$ is

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(a) $\frac{\pi}{4}$ (b) 0 (c) 1 (d) $-\frac{\pi}{4}$

5 If $F(x)=f(x)+f\left(\frac{1}{x}\right)$, where $f(x)=\int_1^x \frac{\log t}{1+t}\,dt$. Then, $F(e)$ is equal to

(a) 1/2 (b) 0 (c) 1 (d) 2

6 The expression $\frac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx}$, where $[x]$ and $\{x\}$ are integral and fractional part of x and $n\in N$, is equal to

(a) $\frac{1}{n-1}$ (b) $\frac{1}{n}$ (c) n (d) $n-1$

7 If $f(x)=\frac{e^x}{1+e^x}$, $I_1=\int_{f(-a)}^{f(a)} xg[x(1-x)]\,dx$ and $I_2=\int_{f(-a)}^{f(a)} g[x(1-x)]\,dx$, then the value of $\frac{I_2}{I_1}$ is

(a) 2 (b) -3 (c) -1 (d) 1

8 The value of $\lim_{x\to\infty}\frac{\{\int_0^x e^{t^2}dt\}^2}{\int_0^x e^{2t^2}dt}$ is

(a) 1/3 (b) 2/3 (c) 1 (d) None of these

9 The integral $\int_0^\pi \sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ is equal to

➜ **JEE Mains 2014**

(a) $\pi-4$ (b) $\frac{2\pi}{3}-4-4\sqrt{3}$
(c) $4\sqrt{3}-4$ (d) $4\sqrt{3}-4-\frac{\pi}{3}$

10 $\int_0^\pi x f(\sin x)\,dx$ is equal to

(a) $\pi\int_0^\pi f(\sin x)\,dx$ (b) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)\,dx$
(c) $\pi\int_0^{\pi/2} f(\cos x)\,dx$ (d) $\pi\int_0^{\pi} f(\cos x)\,dx$

11 If $f(x)=\int_{x^2}^{x^2+1} e^{-t^2}\,dt$, then $f(x)$ increases in

(a) $(2, 2)$ (b) no value of x
(c) $(0,\infty)$ (d) $(-\infty, 0)$

12 $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\dots 3n}{n^{2n}}\right)^{1/n}$ is equal to

➜ **JEE Mains 2016**

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$
(c) $\frac{9}{e^2}$ (d) $3\log 3-2$

13 The least value of the function $f(x)=\int_{5\pi/4}^{x}(3\sin u+4\cos u)\,du$ on the interval $\left[\frac{5\pi}{4},\frac{4\pi}{3}\right]$ is

(a) $\sqrt{3}+\frac{3}{2}$ (b) $-2\sqrt{3}+\frac{3}{2}+\frac{1}{\sqrt{2}}$
(c) $\frac{3}{2}+\frac{1}{\sqrt{2}}$ (d) None of these

14 If $g(x)=\int_0^x f(t)\,dt$, where f is such that, $\frac{1}{2}\le f(t)\le 1$, for $t\in[0,1]$ and $0\le f(t)\le\frac{1}{2}$, for $t\in[1, 2]$. Then, $g(2)$ satisfies the inequality

(a) $-\frac{3}{2}\le g(z)<\frac{1}{2}$ (b) $\frac{1}{2}\le g(2)\le\frac{3}{2}$
(c) $\frac{3}{2}<g(2)\le\frac{5}{2}$ (d) $2<g(2)<4$

15 Let $n\ge 1, n\in z$. The real number $a\in(0, 1)$ that minimizes the integral $\int_0^1 |x^n-a^n|\,dx$ is

(a) $\frac{1}{2}$ (b) 2
(c) 1 (d) $\frac{1}{3}$

16 The value of

$$\lim_{n\to\infty}\left\{\tan\left(\frac{\pi}{2n}\right)\tan\left(\frac{2\pi}{2n}\right)\tan\left(\frac{3\pi}{2n}\right)\dots\tan\left(\frac{n\pi}{2n}\right)\right\}^{1/n}$$ is

(a) 1 (b) 2
(c) 3 (d) Not defined

17 If $f(x)=\dfrac{x-1}{x+1}, f^2(x)=f(x), \dots, f^{k+1}(x)=f\{f^k(x)\}$, $k=1, 2, 3, \dots$ and $g(x)=f^{1998}(x)$, then $\int_{1/e}^{1} g(x)\,dx$ is equal to

(a) 0 (b) 1 (c) −1 (d) e

18 If $f(x)$ is a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and $g(x)$ is a function that satisfies $f(x)+g(x)=x^2$. Then, the value of $\int_0^1 f(x)g(x)\,dx$, is

(a) $e-\dfrac{e^2}{2}-\dfrac{5}{2}$ (b) $e+\dfrac{e^2}{2}-\dfrac{3}{2}$

(c) $e-\dfrac{e^2}{2}-\dfrac{3}{2}$ (d) $e+\dfrac{e^2}{2}+\dfrac{5}{2}$

19 If $n>1$, then

Statement I $\int_0^\infty \dfrac{dx}{1+x^n}=\int_0^1 \dfrac{dx}{(1-x^n)^{1/n}}$

Statement II $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

20 Consider $\sin^6 x$ and $\cos^6 x$ is a periodic function with π.

Statement I $\int_0^{\pi/2}(\sin^6 x+\cos^6 x)\,dx$ lie in the interval $\left(\dfrac{\pi}{8}, \dfrac{\pi}{2}\right)$.

Statement II $\sin^6 x+\cos^6 x$ is periodic with period $\pi/2$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1

1 (b)	**2** (d)	**3** (c)	**4** (b)	**5** (b)	**6** (b)	**7** (d)	**8** (a)	**9** (a)	**10** (c)
11 (a)	**12** (b)	**13** (d)	**14** (d)	**15** (d)	**16** (d)	**17** (b)	**18** (c)	**19** (b)	**20** (c)
21 (b)	**22** (a)	**23** (c)	**24** (b)	**25** (d)	**26** (d)	**27** (c)	**28** (b)	**29** (a)	**30** (a)
31 (c)	**32** (b)	**33** (a)	**34** (c)	**35** (d)	**36** (b)	**37** (a)	**38** (a)	**39** (d)	**40** (b)
41 (c)	**42** (d)	**43** (d)	**44** (d)	**45** (a)					

SESSION 2

1 (d)	**2** (d)	**3** (c)	**4** (a)	**5** (a)	**6** (d)	**7** (a)	**8** (d)	**9** (d)	**10** (c)
11 (d)	**12** (b)	**13** (b)	**14** (b)	**15** (a)	**16** (a)	**17** (c)	**18** (c)	**19** (b)	**20** (b)

Hints and Explanations

SESSION 1

1 Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$

$= \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{1-\cos^2 x} dx$

$= \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{\sin^2 x} dx$

$= \int_{\pi/4}^{3\pi/4} (\text{cosec}^2 x - \text{cosec}\, x \cot x) dx$

$= [-\cot x + \text{cosec}\, x]_{\pi/4}^{3\pi/4}$

$= [(1+\sqrt{2}) - (-1+\sqrt{2})] = 2$

2 Since, f is continuous function.

Let $x = t - 1$

$\therefore$ $dx = dt$

When x tends to -3 and 5, then t tends to -2, 6.

Therefore, $\int_{-3}^{5} f(x)\,dx = \int_{-2}^{6} f(t-1)\,dt$

$= \int_{-2}^{6} f(x-1)\,dx$

3 Let $I = \int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}]\,dx$

$= \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$

$= \sqrt{2} \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\,dx = dt$

$\therefore\ I = \sqrt{2} \int_{-1}^{0} \frac{dt}{\sqrt{1-t^2}} \Rightarrow I = \sqrt{2}\,[\sin^{-1} t]_{-1}^{0}$

$= \sqrt{2}\,[0 - (-\pi/2)] = \frac{\pi}{\sqrt{2}}$

4 $\int_0^1 \sin\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right] dx$

Put $x = \cos\theta$, then

$\sin\left[2\tan^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right]$

$= \sin\left[2\tan^{-1}\left(\cot\frac{\theta}{2}\right)\right]$

$= \sin\left[2\tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}\right]$

$= \sin\left[2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$

$= \sin(\pi - \theta) = \sin\theta$

$= \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$

$\therefore \int_0^1 \sin\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right] dx$

$= \int_0^1 \sqrt{1-x^2}\,dx$

$= \left[\frac{1}{2} x\sqrt{1-x^2}\right]_0^1 + \frac{1}{2}[\sin^{-1} x]_0^1 = \frac{\pi}{4}$

5 $I_{2(n)} - I_{2(n-1)}$

$= \int_0^{\pi/2} \frac{[\sin^2 nx - \sin^2(n-1)x]}{\sin^2 x} dx$

$= \int_0^{\pi/2} \frac{\sin(2n-1)x \cdot \sin x}{\sin^2 x} dx$

$= \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = I_{1(n)}$

$\Rightarrow I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$

6 $\int_{-1}^{1} \{x^2 + x - 3\}\,dx = \int_{-1}^{1} \{x^2 + x\}\,dx$

$= \int_{-1}^{1} (x^2 + x - [x^2 + x])\,dx$

$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} - \int_{-1}^{0} [x^2 + x]\,dx$

$- \int_0^{\frac{\sqrt{5}-1}{2}} [x^2 + x]\,dx$

$- \int_{(\sqrt{5}-1)/2}^{1} [x^2 + x]\,dx$

$= \frac{2}{3} + 1 - 0 - 1\left(1 - \frac{\sqrt{5}-1}{2}\right)$

$= \frac{2}{3} + \frac{\sqrt{5}-1}{2} = \frac{1}{6}(1 + 3\sqrt{5})$

7 $\int_0^2 [x^2]\,dx = \int_0^1 [x^2]\,dx$

$+ \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\,dx$

$+ \int_{\sqrt{3}}^{2} [x^2]\,dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx$

$+ \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx$

$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^{2}$

$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$

$= 5 - \sqrt{3} - \sqrt{2}$

8 $\int_{-2}^{x} |2t|\,dt = f(x)$

$= \int_{-2}^{0} |2t|\,dt + \int_0^x 2t\,dt$

$= -2\left[\frac{t^2}{2}\right]_{-2}^{0} + 2\left[\frac{t^2}{2}\right]_0^x$

$= -2\,[0-2] + 2\left[\frac{x^2}{2} - 0\right] = 4 + x^2$

9 Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

$= \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x} dx$

$+ \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx$ ...(i)

On putting $x = -x$ in 1st integral, we get

$\int_{-\pi/2}^{0} \frac{\cos x}{1+e^x} dx = \int_0^{\pi/2} \frac{e^x \cos x}{1+e^x} dx$

$\therefore\ I = \int_0^{\pi/2} \frac{e^x \cos x}{1+e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx$

$= \int_0^{\pi/2} \frac{(1+e^x)\cos x}{(1+e^x)} dx$

$= \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$

10 $\int_0^3 (3ax^2 + 2bx + c)\,dx$

$= \int_1^3 (3ax^2 + 2bx + c)\,dx$

$\Rightarrow \int_0^1 (3ax^2 + 2bx + c)\,dx$

$+ \int_1^3 (3ax^2 + 2bx + c)\,dx$

$= \int_1^3 (3ax^2 + 2bx + c)\,dx$

$\Rightarrow \int_0^1 (3ax^2 + 2bx + c)\,dx = 0$

$\Rightarrow \left[\frac{3ax^3}{3} + \frac{2bx^2}{2} + cx\right]_0^1 = 0$

$\therefore\ a + b + c = 0$

11 Since, $\int_1^a [x] f'(x)\,dx = \int_1^2 f'(x)\,dx$

$+ \int_2^3 2f'(x)\,dx + \ldots + \int_{[a]}^{a} [a] f'(x)\,dx$

$= [f(x)]_1^2 + 2\,[f(x)]_2^3 + \ldots + [a][f(x)]_{[a]}^{a}$

$= f(2) - f(1) + 2f(3) - 2f(2) + \ldots$
$+ [a]\,f(a) - [a]\,f([a])$

$= [a]\,f(a) - \{f(1) + f(2) + \ldots + f([a])\}$

12 Let $I = \int_0^{\pi/2} \left|\sin\left(x - \frac{\pi}{4}\right)\right| dx$

$= -\int_0^{\pi/4} \sin\left(x - \frac{\pi}{4}\right) dx$

$+ \int_{\pi/4}^{\pi/2} \sin\left(x - \frac{\pi}{4}\right) dx$

$= \left[\cos\left(x - \frac{\pi}{4}\right)\right]_0^{\pi/4} - \left[\cos\left(x - \frac{\pi}{4}\right)\right]_{\pi/4}^{\pi/2}$

$= 1 - \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} - 1\right) = 2 - \sqrt{2}$

13 $\int_0^{3\pi/2} \sin\left[\frac{2x}{\pi}\right] dx = \int_0^{\pi/2} \sin\left[\frac{2x}{\pi}\right] dx$

$+ \int_{\pi/2}^{\pi} \sin\left[\frac{2x}{\pi}\right] dx + \int_{\pi}^{3\pi/2} \sin\left[\frac{2x}{\pi}\right] dx$

$= 0 + \sin 1 \int_{\pi/2}^{\pi} dx + \sin 2 \int_{\pi}^{3\pi/2} dx$

$= \frac{\pi}{2}(\sin 1 + \sin 2)$

14 Here, $\int_0^{1.5} x[x^2]\,dx$

$I = \int_0^1 x \cdot 0\,dx + \int_1^{\sqrt{2}} x \cdot 1\,dx + \int_{\sqrt{2}}^{1.5} x \cdot 2dx$

$$= 0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5}$$

$$= \frac{1}{2}\{2-1\} + \{(1.5)^2 - 2\}$$

$$= \frac{1}{2} + 2.25 - 2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

15 Let $I = \int_0^1 \frac{8 \log (1+x)}{(1+x^2)} dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

When $x = 0 \Rightarrow \tan\theta = 0$

$\therefore \quad \theta = 0$

When $x = 1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \frac{8 \log (1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta d\theta$$

$$= 8\int_0^{\pi/4} \log (1+\tan\theta) d\theta$$

$$I = 8\int_0^{\pi/4} \log (1+\tan\theta) d\theta \qquad \text{...(i)}$$

$$= 8 \cdot \left(\frac{\pi}{2} \cdot \log 2\right)$$

$$\left[\because \int_0^{\pi/4} \log (1+\tan\theta) d\theta = \frac{\pi}{8} \log 2\right]$$

$$= \pi \log 2$$

16 Let $I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2} dx$$

$[\because \cos x - \sin x > 0$ when $x \in \left(0, \frac{\pi}{4}\right)$ and $\cos x - \sin x < 0$ when $x \in (\pi/4, \pi/2)]$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] + \left[\left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)\right] + \left[-1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \left[\frac{2}{\sqrt{2}} - 1\right] + [-1 + \sqrt{2}]$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2}) = 2(\sqrt{2} - 1)$$

17 $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx = \int_{e^{-1}}^{1} \left|\frac{\log_e x}{x}\right| dx + \int_1^{e^2} \left|\frac{\log_e x}{x}\right| dx$

$$= \int_{e^{-1}}^{1} \frac{-\log_e x}{x} dx + \int_1^{e^2} \frac{\log_e x}{x} dx$$

$$= \int_{-1}^{0} -z\, dz + \int_0^2 z\, dz$$

[put $\log_e x = z \Rightarrow (1/x) dx = dz$]

$$\therefore I = \left[-\frac{z^2}{2}\right]_{-1}^0 + \left[\frac{z^2}{2}\right]_0^2 = \frac{1}{2} + 2 = \frac{5}{2}$$

18 Let $I = \int_0^1 f(k-1+x) dx$

$\Rightarrow \quad I = \int_{k-1}^{k} f(t) dt$, where

$t = k - 1 + x$

$\Rightarrow \quad dt = dx$

$\Rightarrow \quad I = \int_{k-1}^{k} f(x) dx$

$$\therefore \sum_{k=1}^{n} \int_{k-1}^{k} f(x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx + \ldots + \int_{n-1}^{n} f(x) dx$$

$$= \int_0^n f(x) dx$$

19 $\int_{-3}^{5} f(|x|) dx = \int_{-3}^{3} f(|x|) dx + \int_3^5 f(|x|) dx$

$$= 2\int_0^3 f(x) dx + \int_3^5 f(x) dx$$

$$= 2\left(\int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx\right) + \left(\int_3^4 f(x) dx + \int_4^5 f(x) dx\right)$$

$$= 2\left(0 + \frac{1}{2} + \frac{2^2}{2}\right) + \left(\frac{9}{2} + \frac{16}{2}\right) = \frac{35}{2}$$

20 Given that, $I_1 = \int_0^1 2^{x^2} dx$

$I_2 = \int_0^1 2^{x^3} dx,$

$I_3 = \int_1^2 2^{x^2} dx$

and $I_4 = \int_1^2 2^{x^3} dx$

Since, $2^{x^3} < 2^{x^2}$ for $0 < x < 1$

and $\quad 2^{x^3} > 2^{x^2}$ for $x > 1$

$\therefore \quad \int_0^1 2^{x^3} dx < \int_0^1 2^{x^2} dx$

and $\quad \int_1^2 2^{x^3} dx > \int_1^2 2^{x^2} dx$

$\Rightarrow \quad I_2 < I_1$ and $I_4 > I_3$

21 $I = \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x}\, dx$

$$= \int_{7\pi/4}^{2\pi} |\tan x| dx + \int_{2\pi}^{7\pi/3} |\tan x| dx$$

$$= -\int_{7\pi/4}^{2\pi} \tan x\, dx + \int_{2\pi}^{7\pi/3} \tan x\, dx$$

$$= -[\log \sec x]_{7\pi/4}^{2\pi} + [\log \sec x]_{2\pi}^{7\pi/3}$$

$$= -\left[\log \sec 2\pi - \log \sec \frac{7\pi}{4}\right] + \left[\log \sec \frac{7\pi}{3} - \log \sec 2\pi\right]$$

$$= -\left[\log 1 - \log \sec \frac{\pi}{4}\right] + \left[\log \sec \frac{\pi}{3} - \log 1\right]$$

$$= \log\sqrt{2} + \log 2 = \frac{1}{2} \log 2 + \log 2$$

$$= \frac{3}{2} \log 2$$

22 We have, $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx$$

$$\text{Thus, } 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right\}\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log\left\{\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right\}\right]$$

$$= \frac{1}{\sqrt{2}} [\log(\sqrt{2}+1) - \log(\sqrt{2}-1)]$$

$$= \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \frac{1}{\sqrt{2}} \log\left\{\frac{(\sqrt{2}+1)^2}{1}\right\}$$

$$= \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)]$$

$$\therefore \quad I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

23 $\lim_{n\to\infty} \frac{1}{n}\left\{\frac{1}{n+1} + \frac{2}{n+2} + \ldots + \frac{3n}{n+3n}\right\}$

$$= \lim_{n\to\infty} \sum_{r=1}^{3n} \frac{1}{n}\left(\frac{r}{n+r}\right)$$

$$= \int_0^3 \frac{x}{1+x} dx$$

$$= \int_0^3 \left(1 - \frac{1}{1+x}\right) dx$$

$$= [x - \ln(1+x)]_0^3 = 3 - \ln 4$$

$$= 3 - 2\ln 2$$

24 $\because I = \int_0^a f(x) \cdot g(x)\, dx$

$$= \int_0^a f(a-x)\, g(a-x) dx$$

$$= \int_0^a f(x)\{a - g(x)\} dx$$

$$= a\int_0^a f(x) dx - \int_0^a f(x) \cdot g(x) dx$$

$$= a\int_0^a f(x) dx - I$$

$$\therefore \quad I = \frac{a}{2}\int_0^a f(x) dx$$

25 Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2\left(\frac{-\pi}{2} + \frac{\pi}{2} - x\right) dx}{1 + 2^{\frac{-\pi}{2} + \frac{\pi}{2} - x}} dx$$

$$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx\right]$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}}\, dx$$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1+2^x}\, dx$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1+2^x}\, dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} 2\sin^2 x\, dx$$

$$= \int_0^{\pi/2} (1-\cos 2x)\, dx$$

$$\Rightarrow \quad 2I = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

26 Let $\quad I = \int_0^{\pi} [\cot x]\, dx \qquad \text{...(i)}$

$$\Rightarrow \quad I = \int_0^{\pi} [\cot(\pi - x)]\, dx$$

$$= \int_0^{\pi} [-\cot x]\, dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} [\cot x]\, dx + \int_0^{\pi} [-\cot x]\, dx$$

$$= \int_0^{\pi} (-1)\, dx$$

$$\left[\because [x] + [-x] = \begin{cases} -1 \text{ if } x \notin Z \\ 0, \text{if } x \in Z \end{cases}\right]$$

$$= [-x]_0^{\pi} = -\pi$$

$$\therefore I = -\frac{\pi}{2}$$

27 $$I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}\, dx$$

$$= \int_2^4 \frac{2\log x}{2\log x + \log(6-x)^2}\, dx$$

$$= \int_2^4 \frac{2\log x\, dx}{2[\log x + \log(6-x)]}$$

$$\Rightarrow I = \int_2^4 \frac{\log x dx}{[\log x + \log(6-x)]} \qquad \text{...(i)}$$

$$\Rightarrow I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x}\, dx \qquad \text{...(ii)}$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

On adding Eqs.(i) and (ii),we get

$$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)}\, dx$$

$$\Rightarrow \quad 2I = \int_2^4 dx = [x]_2^4$$

$$\Rightarrow \quad 2I = 2 \Rightarrow I = 1$$

28 Let $I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2 x]\, dx$

$$\Rightarrow \quad I = \int_{-3\pi/2}^{-\pi/2} \left[\left(-\frac{\pi}{2} - \frac{3\pi}{2} - x + \pi\right)^3 + \cos^2\left(-\frac{\pi}{2} - \frac{3\pi}{2} - x\right)\right] dx$$

$$\left(\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right)$$

$$\Rightarrow \quad I = \int_{-3\pi/2}^{-\pi/2} [-(x+\pi)^3 + \cos^2 x]\, dx$$

$$\Rightarrow 2I = \int_{-3\pi/2}^{-\pi/2} 2\cos^2 x\, dx$$

$$= \int_{-3\pi/2}^{-\pi/2} (1 + \cos 2x)\, dx$$

$$= \left[x + \frac{\sin 2x}{2}\right]_{-3\pi/2}^{-\pi/2}$$

$$= \left[-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} - \left(-\frac{3\pi}{2} + \frac{\sin(-3\pi)}{2}\right)\right] = \pi$$

$$\therefore \quad I = \frac{\pi}{2}$$

29 Let $\phi(x) = [f(x) + f(-x)]\,[g(x) - g(-x)]$

$$\therefore \ \phi(-x) = [f(-x) + f(x)]\,[g(-x) - g(x)] = -\phi(x)$$

$$\therefore \quad \int_{-\pi}^{\pi} \phi(x)\, dx = 0$$

($\because \phi(x)$ is an odd function)

30 $$S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)\sqrt{r\,(2n+r)}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n\left(1+\frac{r}{n}\right)\sqrt{\frac{r}{n}\left(2+\frac{r}{n}\right)}}$$

$$S = \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}}$$

$$= \int_0^1 \frac{dx}{(1+x)\sqrt{(1+x)^2 - 1}}$$

$$S = [\sec^{-1}(1+x)]_0^1$$

$$= \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}$$

31 $P = \int_0^{3\pi} f(\cos^2 x)\, dx$ and

$$Q = \int_0^{\pi} f(\cos^2 x)\, dx$$

Also, $\quad P = 3\int_0^{\pi} f(\cos^2 x) dx = 3Q$

$$\therefore \quad P - 3Q = 0$$

32 Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\, dx$$

$$\Rightarrow \quad I = 0 + 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx \qquad \text{...(i)}$$

$\left[\because \frac{2x}{1+\cos^2 x}\right.$ is an odd function and $\frac{2x\sin x}{1+\cos^2 x}$ is an even function$\left.\right]$

$$\Rightarrow I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$$

$$\Rightarrow I = 4\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\, dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\, dx$$

$$\Rightarrow I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx - I$$

[from Eq. (i)]

$$\Rightarrow I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Put $\cos x = t$

$$\Rightarrow -\sin x\, dx = dt$$

$$\therefore I = -2\pi \int_1^{-1} \frac{1}{1+t^2}\, dt$$

$$= 2\pi[\tan^{-1} t]_{-1}^{1} = 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= \pi^2$$

33 Using Newton-Leibnitz's formula, we get

$$t^2\,\{f(t^2)\}\left\{\frac{d}{dt}(t)^2\right\} - 0 \cdot f(0)\left\{\frac{d}{dt}(0)\right\} = 2t^4$$

$$\Rightarrow \quad t^2\{f(t^2)\}\,2t = 2t^4$$

$$\Rightarrow \quad f(t^2) = t$$

$$\Rightarrow \quad f\left(\frac{4}{25}\right) = \pm\frac{2}{5} \qquad \left[\text{put } t = \pm\frac{2}{5}\right]$$

$$\therefore \quad f\left(\frac{4}{25}\right) = \frac{2}{5}$$

[neglecting negative sign]

34 We have,

$$f(x) = \frac{1}{x^2}\int_4^x [4t^2 - 2f'(t)]\, dt$$

$$\therefore \quad f'(x) = \frac{1}{x^2}[4x^2 - 2f'(x)] - \frac{2}{x^3}\int_4^x [4t^2 - 2f'(t)]\, dt$$

$$\Rightarrow \quad f'(4) = \frac{1}{16}[64 - 2f'(4)] - 0$$

$$\therefore \quad f'(4) = \frac{32}{9}$$

35 $$\lim_{x\to 0} \frac{\int_0^{x^2} \sin\sqrt{t}\, dt}{x^3} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sin x \cdot 2x}{3x^2}$$

$$= \frac{2}{3}\lim_{x\to 0} \frac{\sin x}{x}$$

$$= \frac{2}{3} \cdot 1 = \frac{2}{3}$$

36 We have, $(1 + x^2) > x^2, \forall\, x$

$$\Rightarrow \quad \sqrt{1+x^2} > x, \ \forall\, x \in (1, 2)$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}} < \frac{1}{x}, \ \forall\, x \in (1, 2)$$

$$\therefore \quad \int_1^2 \frac{dx}{\sqrt{1+x^2}} < \int_1^2 \frac{dx}{x} \Rightarrow I_1 < I_2$$

$$\Rightarrow \quad I_2 > I_1$$

37 We have, $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

By Leibnitz rule, we get

$$1 = \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}} \cdot \sqrt{1+y^2} = y$$

38 $\lim_{x \to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt = \lim_{x \to 2} \frac{\int_6^{f(x)} 4t^3 \, dt}{(x-2)}$ $\left[\text{form } \frac{0}{0}\right]$

[by Leibnitz's rule]

$= \lim_{x \to 2} \frac{4\{f(x)\}^3}{1} f'(x) = 4\{f(2)\}^3 f'(2)$

$= 4 \times (6)^3 \times \frac{1}{48}$

$\left[\because f(2) = 6 \text{ and } f'(2) = \frac{1}{48}, \text{ given}\right]$

$= 18$

39 $\lim_{n \to \infty} \frac{\left[\begin{array}{c}1 + 2^4 + 3^4 \\ + \ldots + n^4\end{array}\right]}{n^5} - \lim_{n \to \infty} \frac{\left[\begin{array}{c}1 + 2^3 + 3^3 \\ + \ldots + n^3\end{array}\right]}{n^5}$

$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^4 - \lim_{n \to \infty} \frac{1}{n} \times \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^3$

$= \int_0^1 x^4 \, dx - \lim_{n \to \infty} \frac{1}{n} \times \int_0^1 x^3 \, dx$

$= \left[\frac{x^5}{5}\right]_0^1 - 0 = \frac{1}{5}$

40 Since, $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow 1 \le 1 + \sin^3 x \le 2$

$$\Rightarrow \frac{1}{\sqrt{2}} \le \frac{1}{\sqrt{1+\sin^3 x}} \le 1$$

$$\Rightarrow \int_0^{\pi/2} \frac{1}{\sqrt{2}} dx \le \int_0^{\pi/2} \frac{dx}{\sqrt{1+\sin^3 x}} \le \int_0^{\pi/2} dx$$

$$\therefore \frac{\pi}{2\sqrt{2}} \le I \le \frac{\pi}{2}$$

41 Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx,$

because in $x \in (0, 1)$, $x > \sin x$

$$\Rightarrow I < \int_0^1 \sqrt{x} \, dx = \frac{2}{3} [x^{3/2}]_0^1$$

$$\Rightarrow I < \frac{2}{3}$$

and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}}$

$$= \int_0^1 x^{-\frac{1}{2}} dx = 2\,[x^{1/2}]_0^1 = 2$$

$\therefore \quad J < 2$

42 Since, $f(x)$ is continuous in [0, 2].

$\therefore \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$= \int_0^1 x^2 \, dx + \int_1^2 \sqrt{x} \, dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^{3/2}}{3/2}\right]_1^2$

$= \frac{1}{3} + \frac{2}{3}(2^{3/2} - 1)$

$= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} = \left(\frac{4\sqrt{2}-1}{3}\right)$

43 Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ ...(i)

$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_{\pi/6}^{\pi/3} dx$

$\Rightarrow \quad 2I = [x]_{\pi/6}^{\pi/3}$

$\Rightarrow \quad I = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{12}$

Hence, Statement I is false but $\int_a^b f(x)dx$ $= \int_a^b f(a+b-x)\,dx$ is a true statement by property of definite integrals.

44 Since, period of $e^{\sin x}$ is 2π.

$\therefore \int_0^{200} e^{\sin x} \, dx \ne 200\lambda$

45 For $0 < x < 1, x > x^2$

$\Rightarrow \quad -x < -x^2 \Rightarrow \quad e^{-x} < e^{-x^2}$

$\Rightarrow \int_0^1 e^{-x} \cos^2 x \, dx < \int_0^1 e^{-x^2} \cos^2 x dx$

If $f(x) \ge g(x)$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

SESSION 2

1 We have, $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$

$\Rightarrow f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$

$I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx$

$= \int_{\sin\theta}^{\operatorname{cosec}\theta} \left\{-\frac{1}{x^2} f\left(\frac{1}{x}\right)\right\} dx$

$= \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\,dt$, where $t = \frac{1}{x}$

$\Rightarrow \quad I = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(t)\,dt = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

2 $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$

$\therefore \quad \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$

Put $x^3 = t$

$\Rightarrow \quad 3x^2 \, dx = dt$

$\therefore \quad f(t) = \int_1^{64} \frac{e^{\sin t}}{t} dt$

$= [f(t)]_1^{64}$

$= f(64) - f(1)$

On comparing, we get

$k = 64$

3 Let $I = \int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\}\,\{g'(x)\}\,dx$

Put $\quad f\{g(x)\} = z$

$\Rightarrow \quad f'\{g(x)\}\, g'(x)\,dx = dz$

When $\quad x = 1$, then $z = f\{g(1)\}$

When $\quad x = 2$, then $z = f\{g(2)\}$

$\therefore \quad I = \int_{f\{g(1)\}}^{f\{g(2)\}} \frac{1}{z} dz = [\log z]_{f\{g(1)\}}^{f\{g(2)\}}$

$\Rightarrow I = \log f\{g(2)\} - \log f\{g(1)\} = 0$

$[\because g(2) = g(1)]$

4 Put $t = \sin^2 z$ in 1st integral and

$t = \cos^2 u$ in 2nd integral, we get

$dt = 2\sin z \, \cos z$ and

$dt = -2\cos u \sin u du$

$\therefore I = \int_0^x z(2\sin z \cos z)\,dz$

$+ \int_{\pi/2}^{x} -u(2\cos u \sin u du)$

$= \int_0^x z \sin 2z \, dz - \int_{\pi/2}^{x} u \sin 2u \, du$

$= \left[-z \cdot \frac{\cos 2z}{2} + \frac{\sin 2z}{4}\right]_0^x$

$- \left[-\frac{u\cos 2u}{2} + \frac{\sin 2u}{4}\right]_{\pi/2}^{x}$

$= \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \{0 + 0\}\right]$

$- \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \left(\frac{\pi}{4} + 0\right)\right]$

$= \frac{\pi}{4}$

5 Since, $\quad f(x) = \int_1^x \frac{\log t}{1+t} dt$

and $F(x) = f(x) + f\left(\frac{1}{x}\right)$

$\therefore \quad F(e) = f(e) + f\left(\frac{1}{e}\right)$

$\Rightarrow \quad F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$

Put $t = \frac{1}{t}$ in second integration

$\therefore F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{-\log t}{1+\frac{1}{t}} d\left(\frac{1}{t}\right)$

$= \int_1^e \frac{\log t}{1+t}\,dt - \int_1^e \frac{t\log t}{(1+t)} \times \left(\frac{-dt}{t^2}\right)$

$= \int_1^e \frac{\log t}{1+t}\,dt + \int_1^e \frac{\log t}{t(1+t)}\,dt$

$= \int_1^e \frac{\log t}{1+t}\,dt + \int_1^e \frac{\log t}{t}\,dt - \int_1^e \frac{\log t}{(1+t)}\,dt$

$= \int_1^e \frac{\log t}{t}\,dt \qquad \left[\because \frac{1}{t(1+t)} = \frac{1}{t} - \frac{1}{t+1}\right]$

$= \left[\frac{(\log t)^2}{2}\right]_1^e$

$= \frac{1}{2}[(\log e)^2 - (\log 1)^2]$

$= \frac{1}{2}$

6 We have, $\int_0^n [x]dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \dots + \int_{n-1}^n (n-1)\,dx$

$= 1(2-1) + 2(3-2) + 3(4-3) + \dots + (n-1)\{n-(n-1)\}$

$= 1 + 2 + 3 + \dots + (n-1)$

$= \frac{n(n-1)}{2}$

and $\int_0^n \{x\}\,dx = \int_0^n (x - [x])\,dx = \frac{n}{2}$

$\therefore \frac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx} = n - 1$

7 Given that, $f(x) = \frac{e^x}{1+e^x}$

$\therefore \quad f(a) = \frac{e^a}{1+e^a}$ ...(i)

and $f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^a}$ ...(ii)

On adding Eqs. (i) and (ii), we get

$f(a) + f(-a) = 1$

$\Rightarrow \quad f(a) = 1 - f(-a)$

Let $\quad f(-a) = t$

$\Rightarrow \quad f(a) = 1 - t$

Now, $I_1 = \int_t^{1-t} xg\,[x(1-x)]\,dx$ ...(iii)

$\left[\because I = \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

$\Rightarrow \quad I_1 = \int_t^{1-t} (1-x)g[x(1-x)]\,dx$ ...(iv)

On adding Eqs. (iii) and (iv), we get

$2I_1 = \int_t^{1-t} g[x(1-x)]\,dx = I_2$ [given]

$\therefore \frac{I_2}{I_1} = 2$

8 $\lim_{x\to\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$

$= \lim_{x\to\infty} \frac{\left(2\int_0^x e^{t^2}\,dt\right)(e^{x^2})}{e^{2x^2}} \qquad \left[\text{form } \frac{0}{0}\right]$

$= \lim_{x\to\infty} \frac{2\int_0^x e^{t^2}\,dt}{e^{x^2}} \left[\text{form } \frac{0}{0}\right]$

$= \lim_{x\to\infty} \frac{2e^{x^2}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{x} = 0$

9 Use the formula,

$|x-a| = \begin{cases} x-a, & x \ge a \\ -(x-a) & x < a \end{cases}$ to break given integral in two parts and then integrate separately.

$\int_0^\pi \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2}\,dx = \int_0^\pi \left|1 - 2\sin\frac{x}{2}\right|dx$

$= \int_0^{\frac{\pi}{3}} \left(1 - 2\sin\frac{x}{2}\right)dx - \int_{\frac{\pi}{3}}^{\pi}\left(1 - 2\sin\frac{x}{2}\right)dx$

$= \left(x + 4\cos\frac{x}{2}\right)_0^{\pi/3} - \left(x + 4\cos\frac{x}{2}\right)_{\frac{\pi}{3}}^{\pi}$

$= 4\sqrt{3} - 4 - \frac{\pi}{3}$

10 Let $\quad I = \int_0^\pi x\,f(\sin x)\,dx$ ...(i)

$\Rightarrow \quad I = \int_0^\pi (\pi - x) f[\sin(\pi - x)]\,dx$

$\Rightarrow \quad I = \int_0^\pi (\pi - x)\,f(\sin x)\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^\pi \pi f(\sin x)\,dx$

$\Rightarrow I = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$ ...(iii)

$\left[\because \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{if } f(2a-x) = f(x) \\ 0 \quad, \text{if } f(2a-x) = -f(x) \end{cases}\right]$

$\Rightarrow I = \pi\int_0^{\pi/2} f(\sin x)\,dx$

Put $\frac{\pi}{2} - x = t \Rightarrow x = \frac{\pi}{2} - t$

Put $dx = -dt$ in Eq. (iii), we get

$I = \frac{\pi}{2}\int_{-\pi/2}^{\pi/2} f(\cos t)\,dt$

$= \frac{\pi}{2}\int_{-\pi/2}^{\pi/2} f(\cos x)\,dx$

$= \pi\int_0^{\pi/2} f(\cos x)\,dx$

[$\because f(\cos x)$ is an even function]

11 On differentiate the given interval by using Newton-Leibnitz formula,

we get $\quad f'(x) = e^{-(x^2+1)^2} \cdot \left\{\frac{d}{dx}(x^2+1)\right\} - e^{-(x^2)^2} \cdot \left\{\frac{d}{dx}(x)^2\right\}$

$= e^{-(x^2+1)^2} \cdot 2x - e^{-(x^2)^2} \cdot 2x$

$= 2xe^{-(x^4+2x^2+1)}(1 - e^{2x^2+1})$

For $f'(x) > 0$,

then $2x(1 - e^{2x^2+1}) > 0$

$\Rightarrow \quad 2x < 0$

$\Rightarrow \quad x < 0$

12 Let $I = \lim_{n\to\infty}\left(\frac{(n+1)\cdot(n+2)\dots(3n)}{n^{2n}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(\frac{(n+1)\cdot(n+2)\dots(n+2n)}{n^{2n}}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\dots\left(\frac{n+2n}{n}\right)\right)^{\frac{1}{n}}$

On taking log on both sides, we get

$\log I = \lim_{n\to\infty}\frac{1}{n}\left[\log\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{2n}{n}\right)\right\}\right]$

$\Rightarrow \quad \log I = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{2n}{n}\right)\right]$

$\Rightarrow \quad \log I = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)$

$\Rightarrow \quad \log I = \int_0^2 \log(1+x)\,dx$

$\Rightarrow \quad \log I = \left[\log(1+x) \cdot x - \int\frac{1}{1+x}\cdot x\,dx\right]_n^2$

$\Rightarrow \log I = [\log(1+x)\cdot x]_0^2 - \int_0^2 \frac{x+1-1}{1+x}\,dx$

$\Rightarrow \log I = 2\cdot\log 3 - \int_0^2\left(1 - \frac{1}{1+x}\right)dx$

$\Rightarrow \quad \log I = 2\cdot\log 3 - [x - \log|1+x|]_0^2$

$\Rightarrow \quad \log I = 2\cdot\log 3 - [2 - \log 3]$

$\Rightarrow \quad \log I = 3\cdot\log 3 - 2$

$\Rightarrow \quad \log I = \log 27 - 2$

$\Rightarrow \quad I = e^{\log 27 - 2} = 27\cdot e^{-2} = \frac{27}{e^2}$

13 We have, $f'(x) = 3\sin x + 4\cos x$

Since, in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, $f'(x) < 0$, so assume the least value at the point $x = \frac{4\pi}{3}$.

Thus,

$f\left[\frac{4\pi}{3}\right] = \int_{5\pi/4}^{4\pi/3}(3\sin u + 4\cos u)\,du$

$= [-3\cos u + 4\sin u]_{5\pi/4}^{4\pi/3}$

$= \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$

14 We have, $g(x) = \int_0^x f(t)\,dt$

$\Rightarrow \quad g(2) = \int_0^2 f(t)\,dt$

$\Rightarrow \quad g(2) = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

We know that, $m \le f(x) \le M$ for $x \in [a, b]$

$\Rightarrow m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$

$\therefore \quad \frac{1}{2} \le f(t) \le 1$, for $t \in [0, 1]$

and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$

$\Rightarrow \quad \frac{1}{2}(1-0) \le \int_0^1 f(t)\,dt \le 1\,(1-0)$

and $0\,(2-1) \le \int_1^2 f(t)\,dt \le \frac{1}{2}(2-1)$

$\Rightarrow \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1$ and $0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$

$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$

$\therefore \quad \frac{1}{2} \le g(2) \le \frac{3}{2}$

15 Let $f(a) = \int_0^1 |x^n - a^n|\,dx$

$= \int_0^a (a^n - x^n)\,dx + \int_a^1 (x^n - a^n)\,dx$

$= \left[a^n x - \frac{x^{n+1}}{n+1}\right]_0^a + \left[\frac{x^{n+1}}{n+1} - a^n \cdot x\right]_a^1$

$= \left[a^{n+1} - \frac{a^{n+1}}{n+1}\right] + \left[\frac{1}{n+1} - a^n - \frac{a^{n+1}}{n+1} + a^{n+1}\right]$

$= 2a^{n+1} - \frac{2a^{n+1}}{n+1} - a^n + \frac{1}{n+1}$

$= 2a^{n+1}\left[\frac{n}{n+1}\right] - a^n + \frac{1}{n+1}$

$\Rightarrow \quad f'(a) = n(2a-1)a^{n-1}$

Thus, only critical point in (0, 1) is $a = 1/2$

Also, $f'(a) < 0$ for $a \in \left(0, \frac{1}{2}\right)$

and $f'(a) > 0$ for $a \in \left(\frac{1}{2}, 1\right)$.

$\therefore \quad f(a)$ is minimum for $a = \frac{1}{2}$.

16 Let $P = \lim_{n \to \infty} \left\{\prod_{r=1}^{n} \tan\left(\frac{r\pi}{2n}\right)\right\}^{1/n}$

$\therefore \quad \ln P = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \ln \tan\left(\frac{r\pi}{2n}\right)$

$= \int_0^1 \ln \tan\left(\frac{\pi x}{2}\right) dx$

$\Rightarrow \quad \ln P = \frac{2}{\pi}\int_0^{\pi/2} \ln \tan x\,dx$ ...(i)

and $\ln P = \frac{2}{\pi}\int_0^{\pi/2} \ln \cot x\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2\ln P = \frac{2}{\pi}\int_0^{\pi/2} (\ln \tan x + \ln \cot x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi/2} \ln 1\,dx = 0$

$\Rightarrow \quad \ln P = 0$

$\therefore \quad P = 1$

17 We have, $f(x) = \frac{x-1}{x+1}$

$\Rightarrow \quad f^2(x) = f[f(x)] = f\left(\frac{x-1}{x+1}\right)$

$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\frac{1}{x}$

$\Rightarrow \quad f^4(x) = f^2[f^2(x)] = f^2\left(-\frac{1}{x}\right)$

$= \frac{-1}{-\frac{1}{x}} = x$

$\therefore \quad g(x) = f^{1998}(x) = f^2 o f^{1996}(x)$

$\Rightarrow \quad g(x) = f^2[f^{1996}(x)]$

$\Rightarrow \quad g(x) = f^2(x)$ [$\because f^{1996}(x) = \{(\underbrace{f^4 o f^4 o f^4 o \ldots f^4}_{499 \text{ times}})\}(x) = x$]

$\Rightarrow \quad g(x) = -\frac{1}{x}$

$\therefore \int_{1/e}^{1} g(x)\,dx = \int_{1/e}^{1}\left(-\frac{1}{x}\right)dx$

$= -[\log_e x]_{1/e}^{1}$

$\Rightarrow \quad \int_{1/e}^{1} g(x)\,dx = -\left[\log_e 1 - \log_e \frac{1}{e}\right]$

$= -[0 + 1] = -1$

18 Given, $f'(x) = f(x)$

and $f(0) = 1$

Let $f(x) = e^x$ ...(i)

Also, $f(x) + g(x) = x^2$

$\Rightarrow \quad g(x) = x^2 - e^x$ ...(ii)

Now, $\int_0^1 f(x)g(x)\,dx$

$= \int_0^1 e^x(x^2 - e^x)\,dx$

[from Eqs. (i) and (ii)]

$= \int_0^1 x^2 e^x\,dx - \int_0^1 e^{2x}\,dx$

$= [x^2 e^x - \int 2xe^x\,dx]_0^1 - \frac{1}{2}[e^{2x}]_0^1$

$= [x^2 e^x - 2xe^x + 2e^x]_0^1 - \frac{1}{2}(e^2 - 1)$

$= [(x^2 - 2x + 2)e^x]_0^1 - \frac{1}{2}e^2 + \frac{1}{2}$

$= [(1 - 2 + 2)e^1 - (0 - 0 + 2)e^0] - \frac{1}{2}e^2 + \frac{1}{2}$

$= e - 2 - \frac{e^2}{2} + \frac{1}{2} = e - \frac{e^2}{2} - \frac{3}{2}$

19 In LHS, put $x^n = \tan^2\theta$

$\Rightarrow \quad nx^{n-1}dx = 2\tan\theta\sec^2\theta\,d\theta$

$\therefore \int_0^\infty \frac{dx}{1+x^n} = \frac{2}{n}\int_0^{\pi/2} \tan^{1-2+2/n}\theta\,d\theta$

$= \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n)-1}\theta\,d\theta$

In RHS, put $x^n = \sin^2\theta$

$\Rightarrow \quad nx^{n-1}dx = 2\sin\theta\cos\theta\,d\theta$

$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{2}{n}\int_0^{\pi/2} \frac{1}{\cos^{2/n}\theta}$

$\sin^{\frac{2}{n}-1}\theta\cos\theta\,d\theta = \frac{2}{n}\int_0^{\pi/2} \tan^{(2/n)-1}\theta\,d\theta$

20 $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x\,(\sin^2 x + \cos^2 x)$

$= 1 - 3\sin^2 x\cos^2 x$

$= 1 - \frac{3}{4}\sin^2 2x$ $\left[\because \text{period} = \frac{\pi}{2}\right]$

So, the least and greatest value of $\sin^6 x + \cos^6 x$ are $\frac{1}{4}$ and 1.

Hence, $\left(\frac{\pi}{2} - 0\right) \times \frac{1}{4} < \int_0^{\pi/2} (\sin^6 x + \cos^6 x)\,dx < \left(\frac{\pi}{2} - 0\right) \times 1$

$\therefore \frac{\pi}{8} < \int_0^{\pi/2} (\sin^6 x + \cos^6 x)\,dx < \frac{\pi}{2}$